

On the uniqueness of solutions to gauge covariant Poisson equations with compact Lie algebras

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ABSTRACT

It is shown, under rather general smoothness conditions on the gauge potential, which takes values in an arbitrary semi-simple compact Lie algebra \mathfrak{g} , that if a (\mathfrak{g} -valued) solution to the gauge covariant Laplace equation exists, which vanishes at spatial infinity, in the cases of 1,2,3,... space dimensions, then the solution is identically zero. This result is also valid if the Lie algebra is merely compact. Consequently, a solution to the gauge covariant Poisson equation is uniquely determined by its asymptotic radial limit at spatial infinity. In the cases of one or two space dimensions a related result is proved, namely that if a solution to the gauge covariant Laplace equation exists, which is unbounded at spatial infinity, but with a certain dimension-dependent condition on the asymptotic growth of its norm, then the solution in question is a covariant constant.

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1 Introduction

Let \mathfrak{g} denote a semi-simple compact Lie algebra and let Φ and \mathcal{F} denote two \mathfrak{g} -valued functions on an n -dimensional Euclidean space R^n . Points in R^n are denoted by $x := (x^1, x^2, \dots, x^n)$. The covariant Poisson equation is then the following,

$$\sum_{k=1}^n \nabla_k(A) \nabla^k(A) \Phi(x) = \mathcal{F}(x), \quad (1)$$

where $\nabla_k(A)$ denotes the following gauge-covariant gradient,

$$\nabla_k(A) := \partial_k + i[A_k, \] \equiv \frac{\partial}{\partial x^k} + i[A_k(x), \]. \quad (2)$$

In the definition (2) above, the symbol $A := (A_1(x), A_2(x), \dots, A_n(x))$ is a \mathfrak{g} -valued gauge potential [1], and the symbol $[\ , \]$ stands for the commutator of \mathfrak{g} -valued quantities. The covariant gradient with an upper index k , $\nabla^k(A)$, is defined as follows,

$$\nabla^k(A) := -\nabla_k(A). \quad (3)$$

Other quantities with upper space indices are defined similarly; *viz.* if V_k is any quantity with a lower (subscript) space index then its counterpart with an upper (superscript) space index V^k is given as follows,

$$V^k := -V_k, \ k = 1, 2, \dots, n. \quad (4)$$

The covariant Poisson equation (1) is a second order elliptic system of partial differential equations which together with appropriate boundary conditions is supposed to determine the quantity $\Phi(x)$, when the inhomogeneous term $\mathcal{F}(x)$ and gauge potential $A(x)$, respectively, are given. These quantities are supposed to satisfy appropriate regularity conditions. We will return to the question of regularity conditions subsequently.

The Lie algebra \mathfrak{g} is specified by the (real-valued) structure constants f_{ab}^c in the commutation relations of the Lie algebra generators T_a , $a = 1, 2, \dots, \dim \mathfrak{g}$,

$$[T_a, T_b] = i f_{ab}^c T_c, \ a, b = 1, 2, \dots, \dim \mathfrak{g}. \quad (5)$$

Here and in what follows, summation of repeated Lie algebra indices, i.e. letters from the beginning of the latin alphabet, over the range $1, 2, \dots, \dim \mathfrak{g}$ is implied, unless otherwise stated.

From now on we will identify the Lie algebra generators with some specific Hermitean matrix representatives. For the purposes in this paper it does not matter which representation is chosen; in physical applications the choice of representations is related to the question of what types of other fields are coupled to the gauge fields, a question which is not our concern here.

Any \mathfrak{g} -valued quantity is in the linear span of the generators T_a , thus

$$\Phi(x) = \Phi^a(x) T_a, \ \mathcal{F}(x) = \mathcal{F}^a(x) T_a, \ A_k(x) = A_k^a(x) T_a, \quad (6)$$

where all the components Φ^a , \mathcal{F}^a and A_k^a are *real*.

The covariant Poisson equation (1) can naturally be written as a system of partial differential equations for the real-valued components Φ^a . Using the commutator algebra (5) one readily obtains the following system of equations,

$$\partial_k \partial^k \Phi^a - 2f_{bc}^a A^{kb} \partial_k \Phi^c - f_{bc}^a (\partial_k A^{kb}) \Phi^c + f_{bd}^a f_{ec}^d A_k^b A^{ke} \Phi^c = \mathcal{F}^a. \quad (7)$$

In the equation above summation over repeated space indices, one lower and one upper, is also implied.

For our purposes the compact matrix notation in Eq. (1) is preferable, and this will be used in what follows.

We will state and prove our main uniqueness theorems in detail for the equation (1) in Sections 3 and 4 below. However before that we collect in the next section a number of facts related to gauge transformations and inner products for semi-simple compact Lie algebras which we need in the proofs.

2 On gauge transformations and Lie algebra inner products

Let G denote the Lie group corresponding to the compact or semi-simple compact Lie algebra defined by the equations (5) and let ω denote a mapping from R^n to G . In the vicinity of the unit element in G any such G -valued element ω can be obtained by exponentiating the Lie algebra, i.e.

$$\omega(x) = e^{i\alpha^a T_a}, \quad (8)$$

where the parameters $\alpha^a(x)$, $a = 1, 2, \dots, \dim \mathfrak{g}$ are sufficiently smooth real-valued functions which vary over a finite range.

A gauge transformation of a gauge potential $A \xrightarrow{\omega} A^\omega$ is then defined as follows for any sufficiently smooth G -valued function $\omega(x)$,

$$A_k(x) \xrightarrow{\omega} A_k^\omega(x) = \omega^{-1}(x) A_k(x) \omega(x) + i(\partial_k \omega^{-1}(x)) \omega(x). \quad (9)$$

Let $U(x)$ be any differentiable \mathfrak{g} -valued function and $\omega(x)$ likewise any differentiable G -valued function. Using Eq. (9) one readily shows that

$$\omega^{-1}(x) (\nabla_k(A) U(x)) \omega(x) = \nabla_k(A^\omega) (\omega^{-1}(x) U(x) \omega(x)). \quad (10)$$

Applying the transformation (10) to the gauge covariant Poisson equation (1) one obtains

$$\sum_{k=1}^n \nabla_k(A^\omega) \nabla^k(A^\omega) (\omega^{-1}(x) \Phi(x) \omega(x)) = \omega^{-1}(x) \mathcal{F}(x) \omega(x), \quad (11)$$

a result which will be used later.

If our Lie algebra is semi-simple and compact we use the following quantities h_{ab} as our Lie algebra metric. The quantities h_{ab} are defined in terms of the structure constants $f_{ab}{}^c$ as follows,

$$h_{ab} = -f_{ab'}{}^{c'} f_{bc'}{}^{b'}. \quad (12)$$

This is the so-called Killing form multiplied with -1 for convenience. It is known [2], that the form (12) is non-degenerate, and furthermore positive definite, if and only if the Lie algebra is semi-simple and compact. The form (12) thus has an inverse, which we denote by h^{ab} ,

$$h^{ab} h_{bc} = \delta_c^a. \quad (13)$$

The form h_{ab} (h^{ab}) is used to lower (raise) Lie algebra indices.

For any two \mathfrak{g} -valued quantities $U = U^a T_a$ and $V = V^a T_a$ we thus define their inner product (U, V) as follows,

$$(U, V) = h_{ab} U^a V^b. \quad (14)$$

The inner product (14) is invariant under the adjoint action of the of the group, i.e.,

$$(U, V) = (\omega^{-1} U \omega, \omega^{-1} V \omega), \forall \omega \in G. \quad (15)$$

The norm $\| \cdot \|_g$ of any Lie algebra valued quantity U is defined in terms of the inner product,

$$\|U\|_g := \sqrt{(U, U)}. \quad (16)$$

We also need the fact that for any three \mathfrak{g} -valued quantities U, V and W the quantity $(U, [V, W])$ is cyclically symmetric,

$$(U, [V, W]) = (V, [W, U]) = (W, [U, V]) \quad (17)$$

The equations (17) follow directly from the fact that the quantities f_{abc} are antisymmetric under the interchange of any indices.

An immediate consequence of Eqs. (17) is the following useful identity,

$$\partial_k(U, V) = (\nabla_k(A)U, V) + (U, \nabla_k(A)V). \quad (18)$$

This identity is valid for any two differentiable \mathfrak{g} -valued quantities U and V and a smooth gauge potential A .

We finally note that one may drop the condition of semi-simplicity above and require only the condition of compactness of the Lie algebra. One is still then guaranteed the existence of an inner product (\cdot, \cdot) in the Lie algebra which is positive definite and satisfies the condition (17) of cyclic symmetry [3]. In the considerations below one only needs these properties of the Lie algebra inner product. The results derived below are therefore also valid for the case of only compact Lie algebras, and not only for the case of *both* semi-simple and compact Lie algebras.

3 Uniqueness theorems in the cases $n = 1$ and $n = 2$

3.1 The one-dimensional case

In the case of one space dimension, the gauge-covariant Poisson equation (1) is fairly trivial. We nevertheless give a brief analysis also of this case for completeness. In one space dimension Eq. (1) becomes the following,

$$\nabla_1(A)\nabla^1(A)\Phi(x) = \mathcal{F}(x), \quad (19)$$

where we now use the notation x in stead of x^1 for simplicity.

We then choose a G -valued quantity ω such that

$$A_1^\omega(x) \equiv \omega^{-1}(x)A_1(x)\omega(x) + i\left(\frac{d}{dx}\omega^{-1}(x)\right)\omega(x) = 0. \quad (20)$$

The condition (20) is equivalent to the differential equation

$$\frac{d}{dx}\omega(x) = -iA_1(x)\omega(x). \quad (21)$$

For any smooth gauge potential $A_1(x)$, the (matrix) differential equation (21) has a G -valued solution, determined apart from a constant initial value at some appropriate point in R^1 . Then, applying a transform of the type given in Eq. (11) to Eq. (19), with the quantity ω determined by Eq. (21), one obtains,

$$\frac{d^2}{dx^2}(\omega^{-1}\Phi\omega) = (\omega^{-1}\mathcal{F}\omega). \quad (22)$$

The question of uniqueness of the solution(s) to Eq. (22) is related to the number of solutions of the homogeneous equation corresponding to Eq. (22) under appropriate boundary conditions. The homogeneous equation in question,

$$\frac{d^2}{dx^2}(\omega^{-1}\Phi_h\omega) = 0, \quad (23)$$

has the general solution

$$\Phi_h(x) = \omega(x)\alpha\omega^{-1}(x)x + \omega(x)\beta\omega^{-1}(x), \quad (24)$$

where α and β are arbitrary constant \mathbf{g} -valued quantities.

If it is required to find solutions $\Phi(x)$ to Eq. (19), which is equivalent to Eq.(22), which grow less rapidly than linearly with x for large x , then the following boundary condition at infinity must hold,

$$\lim_{|x|\rightarrow\infty} \frac{1}{x}||\Phi(x)||_g = 0. \quad (25)$$

Hence the difference of any two such solutions, which satisfies the homogeneous equation (23), and which is of the form Φ_h given in Eq. (24) above, must satisfy the condition

$$\lim_{|x| \rightarrow \infty} \frac{1}{x} \|\Phi_h(x)\|_g = \lim_{|x| \rightarrow \infty} \frac{1}{x} \sqrt{(\alpha, \alpha)x^2 + 2(\alpha, \beta)x + (\beta, \beta)} = 0. \quad (26)$$

Thus one must set $\alpha = 0$ in Eq. (24). Hence, if a solution to Eq. (19) exists, which satisfies the condition (25), then the solution is unique apart from an additive *covariant constant*, i.e. a solution to the homogeneous equation of the form

$$\Phi_h(x) = \omega(x) \beta \omega^{-1}(x). \quad (27)$$

The solution (27) satisfies the condition of covariant constancy,

$$\nabla_1(A)\Phi_h(x) = 0 \quad (28)$$

as a consequence of the condition (21) which determines the quantity ω .

Likewise, if one requires solutions which vanish at infinity, i.e. which satisfy the conditions

$$\lim_{|x| \rightarrow \infty} \|\Phi(x)\|_g = 0, \quad (29)$$

then one must have $\alpha = \beta = 0$ in Eq. (24). Thus, if a solution to Eq. (19) exists, which satisfies the condition (29), then the solution is unique.

We can summarise the discussion above as follows:

Theorem 3.1

Assume that the gauge-covariant Poisson equation in one space dimension, Eq. (19), has a solution in the classic sense, i.e. a solution which is twice continuously differentiable in any finite interval in R^1 , for a given smooth gauge potential $A_1(x)$ and for a given inhomogeneous term $\mathcal{F}(x)$, satisfying appropriate smoothness and asymptotic conditions in R^1 . If such a solution grows less rapidly than linearly with increasing x , i.e. satisfies the condition (25), then the solution is unique apart from an additive covariant constant. Furthermore, if such a solution vanishes asymptotically in the sense of condition (29) then the solution is unique.

3.2 The two-dimensional case

We will now consider the gauge covariant Poisson equation (1) in two-dimensional space R^2 . We assume that this equation has solutions which are twice continuously differentiable in any finite domain in R^2 , for a given smooth gauge potential $(A_1(x^1, x^2), A_2(x^1, x^2))$ and a given inhomogeneous term $\mathcal{F}(x^1, x^2)$, which is supposed to satisfy appropriate smoothness and asymptotic conditions. Let there be two such solutions Φ_A and Φ_B , say. Then their difference

$$\Phi_C := \Phi_A - \Phi_B \quad (30)$$

satisfies the gauge covariant Laplace equation, i.e. the homogeneous equation,

$$\nabla_k(A)\nabla^k(A)\Phi_C(x) = 0, \quad (31)$$

where summation over the space-index k in the range $(1, 2)$ is implied here and in what follows. We then consider the quantity

$$-\nabla^2(\Phi_C, \Phi_C) \equiv \partial_k \partial^k(\Phi_C, \Phi_C). \quad (32)$$

Using the previously established formula (18) and the equation (31) above, one obtains

$$-\nabla^2(\Phi_C, \Phi_C) = 2(\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C). \quad (33)$$

We then express the ∇^2 -operator in terms of polar coordinates (r, θ) in the plane R^2 ,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \quad (34)$$

Integrating the equation (33) over the variable θ in the range $(0, 2\pi)$ one obtains,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \int_0^{2\pi} d\theta (\Phi_C, \Phi_C) = -2 \int_0^{2\pi} d\theta (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C). \quad (35)$$

We now consider the equation (35) above as an ordinary differential equation for the quantity $\int d\theta (\Phi_C, \Phi_C)$ as if the right hand side of the equation were a known quantity. The general solution to this equation is then the following,

$$\begin{aligned} \int_0^{2\pi} d\theta (\Phi_C, \Phi_C) &= \alpha \log r + \beta \\ &- 2(\log r) \int_0^r dr' r' \int_0^{2\pi} d\theta (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C) \\ &+ 2 \int_0^r dr' r' (\log r') \int_0^{2\pi} d\theta (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C), \end{aligned} \quad (36)$$

where α and β are real constants.

In view of the assumed smoothness of the gauge potential A and assumed regularity of the solution Φ_C to the original equation (31), in particular in the vicinity of the origin $r = 0$ in R^2 , one readily concludes that the terms involving integrals in the expression (36) can not generate logarithmic singularities at $r = 0$ which could cancel the term $\alpha \log r$ in that expression. The logarithmic singularity in (36) must be expelled, and this can then only be achieved by setting $\alpha = 0$. Hence, the general solution to the equation (35) when the regularity conditions at $r = 0$ are taken into account, is

$$\int_0^{2\pi} d\theta (\Phi_C, \Phi_C) = \beta - 2(\log r) \int_0^r dr' r' \left(1 - \frac{\log r'}{\log r} \right) \int_0^{2\pi} d\theta (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C), \quad (37)$$

where we now can identify the constant β as follows,

$$\beta = \left[\int_0^{2\pi} d\theta (\Phi_C, \Phi_C) \right]_{r=0} = 2\pi \|\Phi_C(0)\|_g^2. \quad (38)$$

We then consider the asymptotic properties of the solutions Φ_C . Let us impose the condition

$$\lim_{r \rightarrow \infty} \frac{1}{\log r} \int_0^{2\pi} d\theta (\Phi_C, \Phi_C) = 0. \quad (39)$$

Then Eq. (37) implies that

$$\lim_{r \rightarrow \infty} \int_0^r dr' r' \left(1 - \frac{\log r'}{\log r} \right) \int_0^{2\pi} d\theta (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C) = 0. \quad (40)$$

The vanishing of the integral (40) in the limit $r \rightarrow \infty$ implies that the integrand must vanish, since the integrand is non-positive (for $r > 1$) in view of convention (3) and the positive definiteness of the Lie algebra inner product. But then, necessarily,

$$\nabla^k(A)\Phi_C = 0, \quad k = 1, 2, \quad (41)$$

i.e. Φ_C is a covariant constant.

We can thus infer from the result above, that if the gauge covariant Poisson equation in R^2 has a twice continuously differentiable solution Φ , such that (Φ, Φ) is dominated by $\log r$ for large values of r , or more precisely, if the solution satisfies an asymptotic condition of the form (39), then the solution is unique apart from an additive covariant constant. Let us note however, that the existence of a non-zero covariant constant in two (or more) dimensions places certain restrictions on the gauge potential A , which are related to the internal holonomy group [4].

If one requires a more stringent asymptotic condition on the solutions to the gauge covariant Laplace equation than condition (39), namely the condition

$$\lim_{r \rightarrow \infty} \int_0^{2\pi} d\theta (\Phi_C, \Phi_C) \equiv \lim_{r \rightarrow \infty} \int_0^{2\pi} d\theta \|\Phi_C\|_g^2 = 0, \quad (42)$$

then Eqns. (37) and (38) imply as before that Eq. (41) is in force, but also that

$$\left[\int_0^{2\pi} d\theta (\Phi_C, \Phi_C) \right]_{r=0} = 0. \quad (43)$$

Taken together, Eqns. (41) and (43) imply that

$$\Phi_C(x^1, x^2) \equiv 0. \quad (44)$$

We have now demonstrated that the only solution to the gauge covariant Laplace equation in two dimensions which vanishes at spatial infinity in the sense given by Eq. (42), is the identically vanishing solution. This gives rise to a uniqueness theorem for such solutions Φ to the inhomogeneous gauge covariant Poisson equation, which have a given asymptotic behaviour Φ^{as} at spatial infinity, or more precisely, which satisfy a boundary condition of the form

$$\lim_{r \rightarrow \infty} \int_0^{2\pi} d\theta \|\Phi - \Phi^{as}\|_g = 0. \quad (45)$$

Namely, suppose that there exist two solutions, Φ_A and Φ_B , say, to the gauge covariant Poisson equation in two dimensions, which satisfy a boundary condition of the form (45) with some appropriate given asymptotic function Φ^{as} . Then their difference Φ_C satisfies the gauge covariant Laplace equation (31) and the following boundary condition,

$$\begin{aligned}
0 &\leq \lim_{r \rightarrow \infty} \int_0^{2\pi} d\theta \|\Phi_C\|_g^2 = \lim_{r \rightarrow \infty} \int_0^{2\pi} d\theta \|\Phi_A - \Phi_B\|_g^2 \\
&= \lim_{r \rightarrow \infty} \int_0^{2\pi} d\theta \|(\Phi_A - \Phi^{as}) - (\Phi_B - \Phi^{as})\|_g^2 \\
&\leq \lim_{r \rightarrow \infty} \int_0^{2\pi} d\theta (\|\Phi_A - \Phi^{as}\|_g + \|\Phi_B - \Phi^{as}\|_g)^2 = 0.
\end{aligned} \tag{46}$$

But then, in accordance with the reasoning above, Eq. (44) must be in force, i.e. $\Phi_A \equiv \Phi_B$. Thus, if a solution to the gauge covariant Poisson equation in R^2 exists which has a given asymptotic limit in the sense given in Eq. (45), then the solution is unique.

We summarise the results obtained above on the gauge covariant Poisson equation on R^2 as *Theorem 3.2.* below.

Theorem 3.2

Assume that the gauge-covariant Poisson equation in two space dimensions has a solution in the classic sense, i.e. a solution which is twice continuously differentiable in any finite domain in R^2 , for a given smooth gauge potential $(A_1(x^1, x^2), A_2(x^1, x^2))$ and for a given inhomogeneous term $\mathcal{F}(x^1, x^2)$, satisfying appropriate smoothness and asymptotic conditions in R^2 .

If the squared norm $\|\Phi\|_g^2$ of such a solution grows less rapidly than logarithmically with increasing distance from the origin in R^2 , i.e. satisfies the condition (39), then the solution is unique apart from an additive covariant constant.

Furthermore, if such a solution has a given asymptotic radial limit Φ^{as} in the sense of condition (45), then the solution is unique.

4 Uniqueness theorem in the cases $n \geq 3$

The analysis of the uniqueness of solutions to the gauge covariant Poisson equation (1) in the cases $n \geq 3$ proceeds much in the same way as in the two-dimensional case considered in the previous subsection. Thus we assume that the equation (1) has at least two solutions $\Phi_A(x)$ and $\Phi_B(x)$, say, which are twice continuously differentiable in any finite domain in R^n , for any given sufficiently smooth gauge potential $(A_1(x), A_2(x), \dots, A_n(x))$ and given inhomogeneous term $\mathcal{F}(x)$, which satisfies appropriate smoothness and asymptotic conditions in R^n . We consider the difference $\Phi_C(x)$ of these solutions,

$$\Phi_C(x) = \Phi_A(x) - \Phi_B(x). \tag{47}$$

The function $\Phi_C(x)$ satisfies the gauge covariant Laplace equation,

$$\sum_{k=1}^n \nabla_k(A) \nabla^k(A) \Phi_C = 0 \quad (48)$$

as well as such regularity and asymptotic conditions which follow from those conditions of a similar nature which are supposed to be valid for the solutions $\Phi_A(x)$ and $\Phi_B(x)$. In the equation (48) we have for clarity reinstated the explicit summation over the space index k and continue to use this notation below.

Again, using the previously established formula (18) and the equation (48) above, one obtains

$$\nabla^2(\Phi_C, \Phi_C) = -2 \sum_{k=1}^n (\nabla_k(A) \Phi_C, \nabla^k(A) \Phi_C). \quad (49)$$

We then use spherical coordinates [5] on R^n ,

$$\begin{aligned} x^1 &= r \sin \theta_{n-1} \dots \sin \theta_2 \sin \theta_1 \\ x^2 &= r \sin \theta_{n-1} \dots \sin \theta_2 \cos \theta_1 \\ &\vdots \\ x^{n-1} &= r \sin \theta_{n-1} \cos \theta_{n-2} \\ x^n &= r \cos \theta_{n-1} \end{aligned} \quad (50)$$

where

$$r \geq 0, \quad 0 \leq \theta_1 < 2\pi, \quad 0 \leq \theta_k \leq \pi, \quad k \neq 1. \quad (51)$$

The Laplace operator ∇^2 expressed in terms of the spherical coordinates above is as follows,

$$\begin{aligned} \nabla^2 &= \frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} \\ &+ \frac{1}{r^2 \sin^{n-2} \theta_{n-1}} \frac{\partial}{\partial \theta_{n-1}} \sin^{n-2} \theta_{n-1} \frac{\partial}{\partial \theta_{n-1}} \\ &+ \frac{1}{r^2 \sin^2 \theta_{n-1} \sin^{n-3} \theta_{n-2}} \frac{\partial}{\partial \theta_{n-2}} \sin^{n-3} \theta_{n-2} \frac{\partial}{\partial \theta_{n-2}} \\ &+ \dots + \frac{1}{r^2 \sin^2 \theta_{n-1} \dots \sin^2 \theta_2} \frac{\partial^2}{\partial \theta_1^2}. \end{aligned} \quad (52)$$

We still need the invariant (normalised) measure $d\Omega_n$ on the sphere S^{n-1} in terms of the spherical coordinates above,

$$d\Omega_n = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \sin^{n-2} \theta_{n-1} \dots \sin \theta_2 d\theta_1 \dots d\theta_{n-1}. \quad (53)$$

Using the formulae (52) and (53) above, one obtains the following result from Eq. (49),

$$\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} \int d\Omega_n (\Phi_C, \Phi_C) \right) = -2 \sum_{k=1}^n \int d\Omega_n (\nabla_k(A) \Phi_C, \nabla^k(A) \Phi_C), \quad (54)$$

where the integration over the angular variables $\theta_k, k = 1, 2, \dots, n-1$ is over the complete range specified in (51) above. The equation (54) is now considered as an ordinary differential equation for the quantity $\int d\Omega_n (\Phi_C, \Phi_C)$, under the assumption that the right hand side of Eq. (54) is a known quantity. The general solution to Eq. (54) is then the following,

$$\begin{aligned} \int d\Omega_n (\Phi_C, \Phi_C) &= \alpha r^{2-n} + \beta + \\ &+ \frac{2r^{2-n}}{n-2} \int_0^r dr' r'^{n-1} \int d\Omega_n \sum_{k=1}^n (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C) \\ &- \frac{2}{n-2} \int_0^r dr' r' \int d\Omega_n \sum_{k=1}^n (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C), \end{aligned} \quad (55)$$

where α and β are so far undetermined real constants.

We now recall that the solutions Φ_C are supposed to be twice differentiable in any finite domain in R^n , in particular in the vicinity of the origin $r = 0$ in R^n . Moreover, the gauge potential A is supposed to be smooth, in particular near $r = 0$ in R^n . From these conditions follow the estimates below, valid near $r = 0$,

$$\left| \int_0^r dr' r'^{n-1} \int d\Omega_n \sum_{k=1}^n (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C) \right| = O(r^n) \quad (56)$$

and

$$\left| \int_0^r dr' r' \int d\Omega_n \sum_{k=1}^n (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C) \right| = O(r^2). \quad (57)$$

From the estimates (56), (57) and the equation (55) it then follows that

$$\int d\Omega_n (\Phi_C, \Phi_C) = \alpha r^{2-n} + \beta + O(r^2) \quad (58)$$

in the vicinity of $r = 0$. But the solution Φ_C is supposed to be regular, in particular near $r = 0$. The singularity at $r = 0$ which appears to be present in the general solution (55) must be made to disappear, and this can only happen if

$$\alpha = 0 \quad (59)$$

in Eq. (55), in accordance with the estimate (58). Then it also follows that

$$\beta = \left[\int d\Omega_n (\Phi_C, \Phi_C) \right]_{r=0} = \|\Phi_C(0)\|_g^2. \quad (60)$$

Using the conditions (59) and (60), one then finally obtains the following result,

$$\begin{aligned} \int d\Omega_n (\Phi_C, \Phi_C) &= \|\Phi_C(0)\|_g^2 \\ &- \frac{2}{n-2} \int_0^r dr' r' \left(1 - \left(\frac{r'}{r} \right)^{n-2} \right) \int d\Omega_n \sum_{k=1}^n (\nabla_k(A)\Phi_C, \nabla^k(A)\Phi_C). \end{aligned} \quad (61)$$

Let us emphasize that the equation (61) is a relation which is valid for any twice differentiable solution Φ_C to the gauge covariant Laplace equation (48) with a smooth gauge potential A .

Assume now the following boundary condition at spatial infinity for the solution Φ_C to the gauge covariant Laplace equation (48),

$$\lim_{r \rightarrow \infty} \int d\Omega_n (\Phi_C, \Phi_C) = 0. \quad (62)$$

In view of the convention (3) and the positive definiteness of the inner product $(\ , \)$, the condition (62) and the relation (61) together imply that

$$\nabla^k(A)\Phi_C = 0, \quad k = 1, 2, \dots, n, \quad (63)$$

and that

$$||\Phi_C(0)||_g^2 = 0. \quad (64)$$

But the conditions (63) and (64) then finally imply that

$$\Phi_C \equiv 0. \quad (65)$$

We have thus shown that the only twice differentiable solution Φ_C to the gauge covariant Laplace equation (48) with a smooth gauge potential A , which vanishes at spatial infinity in the sense of the condition (62), is the identically vanishing solution (65).

The result above gives rise to a uniqueness theorem for the solutions to the gauge covariant Poisson equation (1) in a space R^n of $n \geq 3$ dimensions, just as in the two-dimensional case. Namely, consider the following boundary conditions,

$$\lim_{r \rightarrow \infty} \int d\Omega_n ||\Phi - \Phi^{as}||_g = 0. \quad (66)$$

As has essentially already been demonstrated in the two-dimensional case, then the difference Φ_C , Eq. (47), of any two solutions Φ_A and Φ_B satisfying the asymptotic condition (66), vanishes at spatial infinity in the sense of Eq. (62). The difference in question also satisfies the gauge covariant Laplace equation (48), and vanishes therefore identically, as has just been demonstrated above. Hence the solution to the gauge covariant Poisson equation (1) is unique if one imposes the boundary conditions (66).

We summarise the result above as the following theorem of uniqueness:

Theorem 4.

Assume that the gauge-covariant Poisson equation (1) in $n \geq 3$ space dimensions has a solution in the classic sense, i.e. a solution which is twice continuously differentiable in any finite domain in R^n , for a given smooth gauge potential $(A_1(x), A_2(x), \dots, A_n(x))$ and for a given inhomogeneous term $\mathcal{F}(x)$, satisfying appropriate smoothness and asymptotic conditions in R^n . If, furthermore, such a solution has a given asymptotic limit Φ^{as} in the sense of condition (66), then the solution is *unique*.

5 Summary and discussion

In this paper a uniqueness theorem has been proved for the gauge covariant Poisson equation in n -dimensional space R^n . The theorem has been obtained by considering the homogeneous counterpart of the Poisson equation in question, i.e. the gauge covariant Laplace equation. It has been shown that the only solution to the gauge covariant Laplace equation, which is twice continuously differentiable in any finite domain in R^n , and which vanishes at infinity is the zero solution. This then proves the uniqueness of that solution of the corresponding Poisson equation, which satisfies appropriate conditions of regularity and has a given asymptotic radial limit.

In one- or two dimensions the asymptotic conditions can be relaxed; it has been shown that in these cases the solutions, which may be unbounded at infinity, are unique apart from an additive covariant constant, even if one does not specify the asymptotic behaviour of the solutions, but merely imposes certain specific limitations on the asymptotic growth of the solutions.

We have not touched upon the question of existence of solutions in this paper. Such questions have been analysed in depth in the three-dimensional case in a recent paper by Salmela [6], who uses modern functional analytic methods in his study. This paper also contains references to physical applications of the gauge covariant Poisson equations in R^3 .

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